# The Lorentz Extension as Consequence of the Family Symmetry

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In this paper we postulate an algebraic model to explain how the symmetry of three lepton species plays its role in the Lorentz extension. Inspired by the two-to-one mapping between the group $SL(2,$ C) and the Lorentz group, we design a mapping between $SL(3,$ C) group, which displays the family symmetry, and a generalized Lorentz group. Following the conventional method, we apply the mapping results to Dirac equation to discuss its transformation invariance, and it turns out that only when the vertex matrix $\gamma_\mu$ is extended to the combination $c_1\gamma_\mu+c_2\gamma_5\gamma_\mu$ can the Dirac-equation-form be reserved. At the same time we find that the Lorentz group has to be extended with an additional generator $\gamma_5$. The generalized vertex matrix is helpful in understanding the axial-like form of weak interaction and the neutrino oscillations.

## I. INTRODUCTION

One mystery of particle physics is that the families of quarks and leptons are just three. Many attempts [1-7] have been made to explain the origin of the families, most of which [1-6] start from a larger group and generate the spinors (families) via the decomposition and breakdown of the group. In Ref. [7, 8] a group $SU(3)_F$ (or alternatively, $S_3$) is assumed to govern the underlying family symmetry. Moreover it is elevated to be gauge symmetry as an extension of the unification symmetry, which could be broken only at or above unification scale $m_U = 10^{19}$GeV.

It is an empirical fact that the unitary groups are responsible for the symmetries of fundamental interactions. And the Lorentz group is another important sort of group which governs the transformation of spinors in quantum fields. If the group governing the family symmetry does indeed rely on the extension of unitary group, then a new kind of interaction has to be introduced. However, there are no clues of its existence up to date, meanwhile the Lorentz extension (violation) in weak-interaction sector is not yet ruled out by experiments. To accept family symmetry as a physical fact, in this paper we adopt the group $SL(3,$ C), which can be graded by $SU(3)_F$ (the concept of "grading" will be explained later), as the underlying symmetry of family and design a mapping to map it to an *extended Lorentz group.*

Many works [9-12] have been done on the possibility that the Lorentz symmetry can also be broken at unifying scale. In Ref. [9] the relationship between neutrino oscillations and Lorentz violation was constructed. Here we suppose the three subgroups of $SL(3$,C), which can be graded by three $SU(2)$ group respectively, form three *Lorentz zones*, because every subgroup can be mapped to Lorentz group. Each sort of neutrinos belongs to a Lorentz zone. Through this way we present a picture based on which the neutrinos oscillate between zones. Since there exists only one Lorentz zone (group) in nature, the three zones must be compacted into one by some manner. We construct the manner by a mapping. The mapping produces the degeneration of the zones as well as the Lorentz violation (Lorentz extension—It turns out that the Lorentz group is broken only by adding one more generator), in coincidence with the hypothesis in Ref. [9]—the Lorentz violation produces the neutrinos' oscillation.

Since the Standard Model successfully describes most low energy particle physics, it is reasonably assumed that the above symmetries' breaking must have some effects at low energies in terms of effective quantum field theories encompassing Standard Model. The effects are expected to be detectable in some existing experiments. In this paper we try to find a way relating the family symmetry to the extended Lorentz symmetry *in the low energy limit.* To do so we directly adopt *the SL(3,C) symmetry* as the family symmetry and at the same time propose a mapping to associate it with the extended Lorentz group. The mapping is analogous to that between the group $SL(2,$ C) and the Lorentz group. We divide the $SL(3$,C) group into three correlative $SL(2$,C) subgroups by their grading manners and find it able to be mapped into three Lorentz groups. While making one $SL(2,$ C) group as *mapping basis* (the concept "*mapping basis*" will be elucidated later), it is found that only one more generator appears to the original Lorentz group. In such a way the Lorentz group is extended.

The remainder of the paper is arranged as follows: in the next section, we briefly review the mapping between the group $SL(2,$ C) and Lorentz group. In the third section, we propose a similar mapping between the group $SL(3,$ C) and extended Lorentz group. In the fourth section we apply the mapping results to Dirac Equation. Finally the concluding remarks are presented.

## II. BRIEF REVIEW OF THE RELATIONSHIP BETWEEN THE GROUP *SL* (2, C) AND THE LORENTZ GROUP

Each element of group $SL(2,\, \mathrm{C})$ has the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

where $a$, $b$, $c$, $d$ are complex numbers and $\det g = ad - bc = 1$, which means $Re(\det g) = 1$ and $Im(\det g) = 0$ and thus leaves the degree of freedom of $g$ to be 6. Equivalently Eq. (1) can be expressed in the form

$$g = g_\mu \sigma^\mu, \tag{2}$$

where $g_\mu (\mu = 0, 1, 2, 3)$ are complex numbers, and

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

We introduce a matrix form $Q$ related to coordinates of space-time $x_\mu = (x_0, \vec{x})$ as follows

$$Q = x_\mu \sigma^\mu, \tag{4}$$

obviously it gives

$$\det Q = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{5}$$

We further define the transformation of $Q$ like

$$Q' = gQg^\dagger, \tag{6}$$

where $Q' = x'_\mu \sigma^\mu$, and $x_\mu$ transforms according to special relativity,

$$x'_\mu = \Lambda^\alpha{}_\mu x_\alpha. \tag{7}$$

From Eqs. (5), (7) we conclude $\det Q (= \det Q')$ an invariant quantity. From Eqs. (4), (6), the transformation of $x_\mu$ yields

$$\begin{array}{l} x'^\alpha = \delta^\alpha{}_\beta x'^\beta = \frac{1}{2} Tr(\sigma^\alpha \sigma^\beta) x'_\beta = \frac{1}{2} Tr(\sigma^\alpha Q') \\ = \frac{1}{2} Tr(\sigma^\alpha g Q g^\dagger) = \frac{1}{2} Tr(\sigma^\alpha g \sigma^\beta g^\dagger) x_\beta \end{array}. \tag{8}$$

Then comparing Eq. (8) with the transformation Eq. (7), we can obtain the mapping between the Lorentz group and the elements of $SL(2,\, \mathrm{C})$,

$$\Lambda^{\alpha\beta} = \frac{1}{2} Tr(\sigma^\alpha g \sigma^\beta g^\dagger), \tag{9}$$

for example, $\Lambda^{00} = |g_0|^2 + \sum\limits_{k=1}^{3} |g_k|^2$. The reverse of the expression (9) yields [13]

$$g = g_0 \sigma^0 + \sum_{k=1}^{3} g_k \sigma_k = D^{-1} [Tr \sigma^0 + \sum_{k=1}^{3} \Lambda_0^k + \Lambda_k^0 - i\varepsilon_\lambda^{0k\rho} \Lambda_\rho^\lambda] \sigma^k, \tag{10}$$

where $D^2 = 4 - Tr\Lambda^2 + (Tr\Lambda)^2 - i\varepsilon_{\rho\tau}^{\mu\lambda} \Lambda_\lambda^\tau \Lambda_\mu^\rho$.

According to Eqs. (9), (10), the following Lorentz transformations and elements of $SL(2,\,\mathrm{C})$ can be mutually carried out:

$$
\begin{array}{cc}
\text{elements of SL (2, C)} & \text{Lorentz transformations} \\
\pm\begin{pmatrix} \cosh\frac{\kappa}{2} & \sinh\frac{\kappa}{2} \\ \sinh\frac{\kappa}{2} & \cosh\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} \cosh\kappa & \sinh\kappa & 0 & 0 \\ \sinh\kappa & \cosh\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\pm\begin{pmatrix} \cosh\frac{\kappa}{2} & i\sinh\frac{\kappa}{2} \\ -i\sinh\frac{\kappa}{2} & \cosh\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} \cosh\kappa & 0 & \sinh\kappa & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\kappa & 0 & \cosh\kappa & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\pm\begin{pmatrix} \exp[\frac{\kappa}{2}] & 0 \\ 0 & \exp[-\frac{\kappa}{2}] \end{pmatrix} & \begin{pmatrix} \cosh\kappa & 0 & 0 & \sinh\kappa \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\kappa & 0 & 0 & \cosh\kappa \end{pmatrix} \\
\pm\begin{pmatrix} \cos\frac{\kappa}{2} & i\sin\frac{\kappa}{2} \\ i\sin\frac{\kappa}{2} & \cos\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\kappa & -\sin\kappa \\ 0 & 0 & \sin\kappa & \cos\kappa \end{pmatrix} \\
\pm\begin{pmatrix} \cos\frac{\kappa}{2} & -\sin\frac{\kappa}{2} \\ \sin\frac{\kappa}{2} & \cos\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\kappa & 0 & \sin\kappa \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\kappa & 0 & \cos\kappa \end{pmatrix} \\
\pm\begin{pmatrix} \exp[i\frac{\kappa}{2}] & 0 \\ 0 & \exp[-i\frac{\kappa}{2}] \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\kappa & -\sin\kappa & 0 \\ 0 & \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array} \tag{11}
$$

where the first three lines represent the Lorentz boost, and last three lines represent the Lorentz rotation, and $\kappa$ stands for the rapidity. Notice that all the elements in the first column are independent and can be mutually expressed with the form in Eq. (2). Therefore the two columns can be seen as the *generating elements* of the $SL(2,\,\mathrm{C})$ and Lorentz group. We can have the generators for the two groups by performing the derivatives with respect to $\kappa$ on these matrices, then making $\kappa = 0$.

## III. THE RELATIONSHIP BETWEEN GROUP *SL*(3,C) AND THE LORENTZ GROUP

A grading $\Xi$ of a Lie algebra $L$ is a decomposition of $L$ into a direct sum of non-zero grading subspaces $L_j \subseteq L$, i.e. $\Xi$:$L = \oplus_{j\in J} L_j$, such that, for each pair $j, k$ of indices from the index set $J$, there exists $l \in J$ with the property $[L_j, L_k] \in L_l$. In preceding section we have seen that the group $SL$(2,C) can be graded by $2\times 2$ Pauli-matrices plus unit matrix $\sigma^0 = I_{2\times 2}$[14]. Now it is known that $SL$(3,C) group has Pauli gradings [14, 15, 16]. And since it is easy to prove that the Pauli grading matrices and the generators of the group $SU(3)$, e.g. Gell-Mann matrices, can be mutually expressed linearly, we recognize that $SL$(3,C) can be graded by generators of the group $SU(3)$ under Gell-Mann representation. The Gell-Mann matrices read

$$
\begin{gathered}
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\end{gathered} \tag{12}
$$

we can divide the SU (3) group into three relevant parts by grouping the generators as follows

$$
\Gamma_1 = \{\lambda_1, \lambda_2, \lambda_3\}, \Gamma_2 = \{\lambda_4, \lambda_5, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\}, \Gamma_3 = \{\lambda_6, \lambda_7, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\}, \tag{13}
$$

where $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ can be derived by appropriately combining $\lambda_3$ and $\lambda_8$. It is obvious that $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ are bases for three $SU(2)$ groups satisfying the commutations of Pauli matrices. Enlightened by the Eqs. (2), (3), adding three matrices $I^1_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $I^2_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $I^3_{3\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$[These three matrices can be derived by appropriately combining $\lambda_3$, $\lambda_8$ and the unit matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.] separately to $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ we have three bases

$$\tilde{\Gamma}_1 = \{I^1_{3\times3}, \lambda_1, \lambda_2, \lambda_3\}, \tilde{\Gamma}_2 = \{I^2_{3\times3}, \lambda_4, \lambda_5, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\}, \tilde{\Gamma}_3 = \{I^3_{3\times3}, \lambda_6, \lambda_7, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\}, \tag{14}$$

which can be mapped to the Lorentz group respectively as in the last section. For example, the Lorentz group mapped from $\tilde{\Gamma}_1$ has the form like Eq. (9)

$$\Lambda^{\alpha\beta} = \frac{1}{2} Tr(\lambda^\alpha g \lambda^\beta g^\dagger), \tag{15}$$

we note that now both the $\lambda^\alpha$, $\lambda^\beta$ and the element $g$ of $SL(2,\text{C})$ group come from the set $\{I^1_{3\times3}, \lambda_1, \lambda_2, \lambda_3\}$. Hereafter we name the $\lambda^\alpha$, $\lambda^\beta$ in Eq. (15) *the basis of the mapping*. From Eq. (15), it can be proved that all of the components $\Lambda^{00}$, $\Lambda^{0i}$, $\Lambda^{i0}$ and $\Lambda^{jk}$ are the same as those from Eq. (9). The similar mappings from $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_3$ to Lorentz group will give the same result. We use these three bases $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_3$ to signify the three families.

According to Eq. (15) and mimicking the matrices in Eq. (11), we can construct the parallelism between the set $\{I^1_{3\times3}, \lambda_1, \lambda_2, \lambda_3\}$ and Lorentz group as follows

$$\begin{array}{cc}
\text{Elements of SL (2, C)} & \text{Lorentz transformations} \\
\pm\begin{pmatrix} \cosh\frac{\kappa}{2} & \sinh\frac{\kappa}{2} & 0 \\ \sinh\frac{\kappa}{2} & \cosh\frac{\kappa}{2} & 0 \\ 0 & 0 & \cosh\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} \cosh\kappa & \sinh\kappa & 0 & 0 \\ \sinh\kappa & \cosh\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\pm\begin{pmatrix} \cosh\frac{\kappa}{2} & i\sinh\frac{\kappa}{2} & 0 \\ -i\sinh\frac{\kappa}{2} & \cosh\frac{\kappa}{2} & 0 \\ 0 & 0 & \cosh\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} \cosh\kappa & 0 & -\sinh\kappa & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\kappa & 0 & \cosh\kappa & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\pm\begin{pmatrix} Exp[\frac{\kappa}{2}] & 0 & 0 \\ 0 & Exp[-\frac{\kappa}{2}] & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} \cosh\kappa & 0 & 0 & \sinh\kappa \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\kappa & 0 & 0 & \cosh\kappa \end{pmatrix} \\
\pm\begin{pmatrix} \cos\frac{\kappa}{2} & i\sin\frac{\kappa}{2} & 0 \\ i\sin\frac{\kappa}{2} & \cos\frac{\kappa}{2} & 0 \\ 0 & 0 & \cos\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\kappa & \sin\kappa \\ 0 & 0 & -\sin\kappa & \cos\kappa \end{pmatrix} \\
\pm\begin{pmatrix} \cos\frac{\kappa}{2} & -\sin\frac{\kappa}{2} & 0 \\ \sin\frac{\kappa}{2} & \cos\frac{\kappa}{2} & 0 \\ 0 & 0 & \cos\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\kappa & 0 & \sin\kappa \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\kappa & 0 & \cos\kappa \end{pmatrix} \\
\pm\begin{pmatrix} Exp[i\frac{\kappa}{2}] & 0 & 0 \\ 0 & Exp[-i\frac{\kappa}{2}] & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\kappa & \sin\kappa & 0 \\ 0 & -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array} \tag{16}$$

where in the left column we have interpreted the *generating elements* of $SL(2,\text{C})$ with $3\times3$ matrices born from the set $\{I^1_{3\times3}, \lambda_1, \lambda_2, \lambda_3\}$, the third diagonal elements are chosen so that when $\kappa = 0$ they reduce to unit matrix. We note that in the second column, despite the different signs in the second, the fourth, and the sixth matrices, all

of them have the reasonable forms like those in Eq. (11), and meanwhile satisfy the condition $\eta_{\alpha\beta}\Lambda^{\alpha\mu}\Lambda^{\beta\nu} = \eta^{\mu\nu}$, where $\eta^{\alpha\beta} = diag(1,-1,-1,-1)$, which is the basic requirement of Lorentz transformation. Subsequently by making derivative to these matrices, one gets the generators of Lorentz group.

The displayed matrices in (16) show up the mapping from $\tilde{\Gamma}_1$ to Lorentz group. In nature, only one Lorentz group should occur. So how the mappings from $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_3$ to Lorentz group manifest their existence is worth studying. Merely in form, we denote the three equivalent Lorentz groups, which are separately produced from $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_3$, by $L_1$, $L_2$, $L_3$. Now we design a mapping that projects $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_3$ also into $L_1$ in an extensive manner. The mapping produces the degeneration of the groups as well as the Lorentz violation. Based on Eq. (15), we choose the *mapping basis* $\lambda^\alpha$, $\lambda^\beta$ still from the set $\{I^1_{3\times3}, \lambda_1, \lambda_2, \lambda_3\}$, but the matrix $g$ from the combination of the elements in set $\tilde{\Gamma}_2$ or $\tilde{\Gamma}_3$. For example, if the matrix $g$ comes from the linear combination of elements in $\tilde{\Gamma}_2$, in analogy with the matrices in (16), the mapping result is as follows

$$
\begin{array}{cc}
\text{Elements of SL (2, C)} & \text{Exended-Lorentz tansformations} \\
\pm\begin{pmatrix} \cosh\frac{\kappa}{2} & 0 & \sinh\frac{\kappa}{2} \\ 0 & \cosh\frac{\kappa}{2} & 0 \\ \sinh\frac{\kappa}{2} & 0 & \cosh\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} \chi & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 \\ 0 & 0 & \chi & 0 \\ 0 & 0 & 0 & \chi \end{pmatrix} \\
\pm\begin{pmatrix} \cosh\frac{\kappa}{2} & 0 & i\sinh\frac{\kappa}{2} \\ 0 & \cosh\frac{\kappa}{2} & 0 \\ -i\sinh\frac{\kappa}{2} & 0 & \cosh\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} \chi & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 \\ 0 & 0 & \chi & 0 \\ 0 & 0 & 0 & \chi \end{pmatrix} \\
\pm\begin{pmatrix} Exp[\frac{\kappa}{2}] & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & Exp[-\frac{\kappa}{2}] \end{pmatrix} & \begin{pmatrix} \frac{1}{2}(1+e^\kappa) & 0 & 0 & \frac{1}{2}(-1+e^\kappa) \\ 0 & e^{\frac{\kappa}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{\kappa}{2}} & 0 \\ \frac{1}{2}(-1+e^\kappa) & 0 & 0 & \frac{1}{2}(1+e^\kappa) \end{pmatrix} \\
\pm\begin{pmatrix} \cos\frac{\kappa}{2} & 0 & i\sin\frac{\kappa}{2} \\ 0 & \cos\frac{\kappa}{2} & 0 \\ i\sin\frac{\kappa}{2} & 0 & \cos\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} \chi & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 \\ 0 & 0 & \chi & 0 \\ 0 & 0 & 0 & \chi \end{pmatrix} \\
\pm\begin{pmatrix} \cos\frac{\kappa}{2} & 0 & -\sin\frac{\kappa}{2} \\ 0 & \cos\frac{\kappa}{2} & 0 \\ \sin\frac{\kappa}{2} & 0 & \cos\frac{\kappa}{2} \end{pmatrix} & \begin{pmatrix} \chi & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 \\ 0 & 0 & \chi & 0 \\ 0 & 0 & 0 & \chi \end{pmatrix} \\
\pm\begin{pmatrix} Exp[i\frac{\kappa}{2}] & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & Exp[-i\frac{\kappa}{2}] \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\kappa & \sin\kappa & 0 \\ 0 & -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
\tag{17}
$$

where $\chi = \cosh^2\frac{\kappa}{2}$.

Almost the same matrices would appear if in Eq. (15) the *mapping basis* still comes from $\tilde{\Gamma}_1$ but the matrix $g$ comes from the linear combination of elements in $\tilde{\Gamma}_3$, with some signs altered. In the column of Exended-Lorentz transformations, we find that after performing derivatives with respect to $\kappa$ and making $\kappa = 0$, only the third matrix leads to a nontrivial generator (the sixth matrix produces a generator of original Lorentz group),

$$
\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{18a}
$$

We call the matrix an extension of Lorentz generators, remembering that $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ is also a generator of original Lorentz group, its effective part can be written as

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{18b}
$$

The Exended-Lorentz matrices due to the mapping from $\tilde{\Gamma}_3$ give the same conclusion. In Eq. (15), if $\lambda^\alpha$, $\lambda^\beta$ is chosen from the set $\tilde{\Gamma}_2$ ($\tilde{\Gamma}_3$), and the matrix $g$ from the combination of the elements in set $\tilde{\Gamma}_1$ or $\tilde{\Gamma}_3$ ($\tilde{\Gamma}_1$ or $\tilde{\Gamma}_2$), we will gain the same results. Without losing generality, in what follows we confine ourselves to developing the results from Eqs. (17), (18). From equation (18b) we can see the structure of extended Lorentz group: the obtained remnant generator commutes with all of the other generators in Lorentz group, the Closure condition is obviously satisfied. We give more discussion on the group structure at the end of next section.

## IV. THE EXTENSION OF KINETIC VERTEX OF DIRAC EQUATION

Free leptons in each of families should satisfy the Dirac equation,

$$\gamma_\mu i\partial^\mu\psi = m\psi, \tag{19}$$

but this equation will not be accurate if the above extension of Lorentz generators is acceptable. We will elucidate the problem and resolve it in this section.

We here define the $\gamma_\mu$ as *kinetic vertex* of the Dirac equation. Performing the Lorentz transformation in Eq. (7) on both sides of Dirac equation and at the same time assuming that $\psi$ transforms according to

$$\psi(x') = S^{-1}\psi(x), \tag{20}$$

where $S$ is a nonsingular $4\times 4$ matrix, then one concludes [17]

$$S^{-1}\gamma_\mu S = \gamma_\nu \Lambda^\nu{}_\mu. \tag{21}$$

Let's introduce the explicit form $\Lambda^\nu{}_\mu = \delta^\nu{}_\mu + \omega^\nu{}_\mu$, where $\omega^\nu{}_\mu$ comes from the generators of Lorentz group and has the expression $\omega^\nu{}_\mu = g^{\nu\lambda}\varepsilon_{\lambda\mu}$[17], and $\varepsilon_{\lambda\mu}$ is an infinitesimal antisymmetric tensor. Substituting this explicit form into Eq. (21) and making some changes, we have

$$\gamma_\mu S - S\gamma_\mu = [\gamma_\mu, S] = S\omega^\nu{}_\mu\gamma_\nu. \tag{22}$$

Furthermore if we take into account the infinitesimal parameter for the transformation $S$

$$S = 1 + \varepsilon_{\mu\nu}S^{\mu\nu}, \tag{23}$$

then to first order, Eq. (22) can be written

$$[\gamma_\mu, \varepsilon_{\rho\sigma}S^{\rho\sigma}] = \omega^\nu{}_\mu\gamma_\nu, \tag{24}$$

accordingly we find the solution of Eq. (24) must be

$$S^{\mu\nu} = \frac{1}{2}\gamma^\mu\gamma^\nu. \tag{25}$$

The above formalism in this section are based on the conventional Lorentz generators. If a generator like matrix (18b) appears, then to reserve the Dirac equation form

$$\Pi_\mu\partial^\mu\psi = m\psi, \tag{26}$$

in which $\Pi_\mu$ is a general form of kinetic vertex being the linear combination of $\gamma$ matrices, its consequence Eq. (24) must be altered correspondingly. Roughly we can rewrite the general form of Eq. (24) analog as

$$[\Pi_\mu, \bar{\varepsilon}_{\rho\sigma}S^{\rho\sigma}] = \bar{\omega}^\nu{}_\mu\Pi_\nu, \tag{27}$$

where $\bar{\omega}^\nu{}_\mu$ now is unit matrix, and $\bar{\varepsilon}_{\mu\nu}$ may not be antisymmetric any longer. Then it is found when

$$\Pi_\mu = \gamma_\mu(1+\gamma_5), \bar{\varepsilon}_{\mu\nu}S^{\mu\nu} = \frac{\bar{\omega}^\nu{}_\mu}{2}\gamma_5, \tag{28}$$

the equation (27) can be satisfied. This means that the *kinetic vertex* of Dirac equation has been extended to include $\gamma_\mu(1+\gamma_5)$, and the Lorentz generators in spinor space $\{\frac{1}{2}\gamma^\mu\gamma^\nu\}$ have been extended to include $\gamma_5$. So now the Dirac equation can be written

$$\gamma_\mu[(1+w) + w\gamma_5]i\partial^\mu\psi = m\psi, \tag{29}$$

where $w$ is a small parameter determined according to specific situation. If in Eq. (29) $\det[(1+w)+w\gamma_5] \neq 0$ and thus $[(1+w)+w\gamma_5]$ has its inverse matrix, the changes appears to kinetic vertex can be transferred to mass term. In that context by making $w$ carry index like $w_F^{\mu\nu}$ or $w_F^{\mu}$($F$ stands for flavor indices to make summation), the discussions in the Ref. [9] can be followed to get the neutrino oscillations as well as mass differences.

We call the set $\{\gamma_5, \{\frac{1}{2}\gamma^\mu\gamma^\nu\}\}$ an extended Lorentz group (in spinor representation): the group Closure condition is kept by recognizing that the product of $\gamma_5$ and any $\frac{1}{2}\gamma^\mu\gamma^\nu$ is still in the set $\{\frac{1}{2}\gamma^\mu\gamma^\nu\}$, and their commutator $[\gamma_5, \frac{1}{2}\gamma_\mu\gamma_\nu] = 0$; $\gamma_5$ and $\{\frac{1}{2}\gamma^\mu\gamma^\nu\}$ actually form a group, in which $\gamma_5$ turns out to be an identity element. And the anti-symmetry of generators' matrix elements of Lorentz group is lost due to the entering of $\gamma_5$.

## V. SUMMARY AND DISCUSSION

In this paper we propose a mapping on the basis of Eq. (15) to elucidate the mutual impacts of three families. The families are thought to have $SL(3, \mathrm{C})$ [which can be graded by $SU(3)_F$] symmetry but show their effect by Lorentz-like invariance. The group $SU(3)_F$ is divided into three $SU(2)$ subgroups, each of which can be mapped into a proper Lorentz group, resembling the way we treating the mapping between group $SL(2, \mathrm{C})$ and Lorentz group. In this way three families are mapped to three equivalent Lorentz groups (*Lorentz zones*). When the mapping from group $SL(3, \mathrm{C})$ to Lorentz group is constructed by choosing one $SL(2, \mathrm{C})$ subgroup as *mapping basis*, the other two subgroups display their impacts only by one more generator additional to that of the original Lorentz group.

If we extend the Dirac equation (19) to include interaction, then it is written

$$\gamma_\mu(i\partial^\mu - gA^\mu)\psi = m\psi. \tag{30}$$

The extension of kinetic vertex in Eq. (29) is obviously applicable to the interaction term $g\gamma_\mu A^\mu$ too,

$$\gamma_\mu A^\mu \to \gamma_\mu[(1+w)+w\gamma_5]A^\mu. \tag{31}$$

The Eq. (31) provides a way the axial-vector interaction $\gamma_\mu\gamma_5 A^\mu$ arises. Reversing the causality, one may form the idea that the appearance of axial-vector form in weak interaction determines the triplicity of lepton families.

At the unification scale, the families' symmetry should be broken to produce mass differences and mass hierarchy of leptons and quarks [8]. The Lorentz invariance might also be broken at this scale to accommodate the inclusion of gravity in a unified theory [9]. So far, by virtue of our mapping, we guess that the two symmetries are possibly dependent on each other at broken scale, as well as dependent on the weak interaction, as argued above. Finally, we stress that the proposed Lorentz extension (Lorentz violation) here should be detected (confirmed or denied) under interactions other than Electrodynamics.

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